Unexpected answers offered by Computer Algebra Systems to school equations

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Abstract

The mathematics curricula of different countries and schools contain different, but certainly significant, amounts of equations. Obviously, the equation types (linear, quadratic and fractional equations, equations that contain an absolute value of an expression, irrational, exponential, logarithmic, trigonometric and literal equations) examined in this paper constitute a large portion of all such equations. The paper focuses on answers offered to different equations by Computer Algebra Systems (here, Axiom, Maxima, Sage, WIRIS and WolframAlpha). Most of the answers are customary for school, but some answers can be somewhat unexpected. The aim of the paper is to classify and map such unexpected answers. 29 phenomena are identified and grouped. The groups are labeled as Input, Form, Unfinished, Domain, Branches, *and* Automatic. *Several phenomena seem to have didactic value for in-depth treatment of certain topics (e.g., equivalence, domain).*

1 Introduction

The history of the use of Computer Algebra Systems (CAS) in learning and teaching school mathematics extends over many years and contains various positive and not so positive moments. The stories of different countries, schools and teachers are quite different. While the use of CAS in school mathematics can be commonplace in some regions, there are other regions where CAS are not used (at least officially). While handheld CAS-calculators (like TI-92+ or Casio ClassPad 300) are popular in some places, CAS-programs (like Derive, Maple or Mathematica) are preferred in other places. In any case, the Internet gives many people (incl. students and teachers) a possibility to use CAS, either online or after downloading. Indeed, the CAS used in the current paper (Axiom, Maxima, Sage, WIRIS, WolframAlpha) are quite easily available on the Internet. There are different ways of using CAS in the learning or teaching of school mathematics. Usually, one would suppose that a CAS gives the correct and expected answer. In many cases, the CAS does indeed give the correct and expected answer. The focus of this paper, however, is on answers that are somewhat unexpected. The aim is not to criticise or rank specific CAS but to identify the phenomena that could be useful for learning and teaching school mathematics.

The mathematical area under consideration in the paper is equation solving. Equation solving constitutes an important part in school mathematics curricula, although there could be differences in emphases and range between different countries or school types. Equations are also important in the use of Computer Algebra Systems. Most answers offered by CAS to equations are customary for school, but there are some answers that would be somewhat unexpected (or incompatible with the teaching practice) in school. (It is notable that expectations can vary in case of different countries or textbooks or teachers. We describe this situation as country-dependence, although the regions of similar expectations do not necessarily have the same borders as countries.) The paper focuses on classification and mapping of such answers. The types of unexpected answers vary in reason, weight and range. We try to discuss the cases that could be useful for didactic purposes.

The mapping includes linear, quadratic and fractional equations, equations that contain an absolute value of an expression, irrational, exponential, logarithmic, trigonometric and literal equations. More than 120 equations are used as test examples. Most of them are school-like equations but some are rather artificial equations, designed to elicit specific behaviour. As CAS are constantly improved and the new versions and new CAS appear, the situation with a particular version of a particular CAS is not overly important. Rather, what could be useful is an understanding of the possible range and nature of the phenomena, especially when a didactic value of an unexpected answer can be capitalised in the introduction or accentuation of several topics. Such topics (e.g. equivalence, domain) can be concealed at first, but are actually important. The scope of this article is restricted to mapping and (preliminary) classification. The ascertainment of the actual importance requires further experiments, involving teachers and students as well.

The first section after the introduction describes related works. The third section introduces the used CAS and commands, and explains what is meant by 'unexpected answer'. Sections 4 and 5 constitute the main parts of the article. The phenomena are identified by the types of equations in section 4. The fifth section is structured by the types of phenomena.

Every CAS has its own style of notation. The user has to be familiar with the notation. However, we do not deal with all notation questions in detail. For the most part, the usual (textbook-style) two-dimensional notation is used in the paper – input and output is 'translated'. Some screenshots are provided as well.

2 Related works

The 'predecessors' of the paper belong to a number of areas. The first important category of works includes reviews of the capabilities of the different CAS. The most extensive of them is [34], also [3], [4] are notable. Unfortunately, these reviews do not focus on school mathematics and include only a few examples from school. Some instances considered in this paper are taken from these lists of examples. However, Wester did not forget school mathematics; he said: *One could invoke* mindset(elementary_math_student) *to initially declare all variables to be real, make* $\sqrt{-1}$ *undefined, etc., for example* ([34]).

The second essential source of this paper includes the papers and books that discuss learning and teaching mathematics with the help of CAS. Many of them present a more general background (e.g.,

[8], [10], [23], [17]). Others are focused on the answers obtained by CAS (as [21]). The articles about the differences between the approaches used in a CAS and in school (as [11]) hit the nail on the head in the context of this paper. The paper [11], based on the experiments of Paul Drijvers, was an important motivator for the study, perhaps not only for this article but further work as well. His work is focused on parameters (incl. equation with parameter), but also proposes some general ideas. There are references to the items on his list of obstacles in the discussion in section 5. The instrumental theory (see [2], [30], [16], [12]) is not explicitly discussed here, but it is present in the background.

The third relevant area of work pertains to teaching and learning of equations (with or without computers). The study [22] gives an overview of differences of equation teaching in different countries. The book [19] discusses future trends in school algebra. This is also important for further study where the importance and usefulness of particular phenomena is under consideration. Teaching and learning of equations is also focused, e.g., in [9], [18], [20].

The author of this paper has written several related articles. Some of them are more specific in a mathematical sense: about infinities and indeterminates ([26]), about equivalence ([24]), about domain ([28]). At the same time, the area of sample exercises encompasses simplifications in addition to equations. The article [27] evaluates branching diversities and is also useful for this paper. The article [29] is dedicated to school equations, but a different list of CAS is used and the present paper provides a more detailed classification of equations.

3 CAS and commands. Unexpected answer

3.1 CAS and commands

There are many Computer Algebra Systems with different versions in the world and the situation is continuously changing. It is not possible to test them all. Different reviewers use different sets of CAS; different articles ([34], [4], [15]) deal with different CAS even in Wester's book [33]. The main criterion for the choice of CAS for the current paper was ease of availability. The paper is based on Computer Algebra Systems Maxima ([**?**]), OpenAxiom ([**?**]), Sage ([**?**]), WIRIS ([**?**]), WolframAlpha ([**?**]) and reflects the situation in the summer 2010. Other CAS, versions and commands could work in a different way. A CAS may have additional options, for example, for showing intermediate steps.

The discussed CAS are quite dissimilar; even the declared types are different: 'Computer Algebra System', 'Open Scientific Computation Platform', 'Computational Knowledge Engine'... Some of them are web-based, some are stand-alone downloadable programmes. Some are products of specific companies; others are open-source projects. For our purposes, it is only important that the CAS should enable the user (student or teacher) to solve equations.

The CAS could have different possibilities for solving equations. The solution process (and the answer) can be sensitive to a change of command, additional arguments, the form of an argument, etc. In this study, we mainly use the most natural method of solution — the command solve. Other commands (like *find_root*) are used rarely and these instances of use are specifically mentioned.

3.2 Unexpected answer

Before starting to discuss unexpected answers, the term itself should be defined. For our purposes, an answer is considered to be unexpected if it differs in some way from the answer that the student/teacher/textbook expects/waits for/presents. In reality, this depends on many circumstances. The expectations could vary depending on the curriculum, the teacher, the textbook, etc. Such unexpected answers are often correct, but according to different standards. A rough classification of CAS answers is presented on the Figure 1. In our study, we focus on reasonable unexpected answers. As we try to discuss didactically useful answers, the terms 'didactical answers' or 'instructive answers' could be appropriate as well.

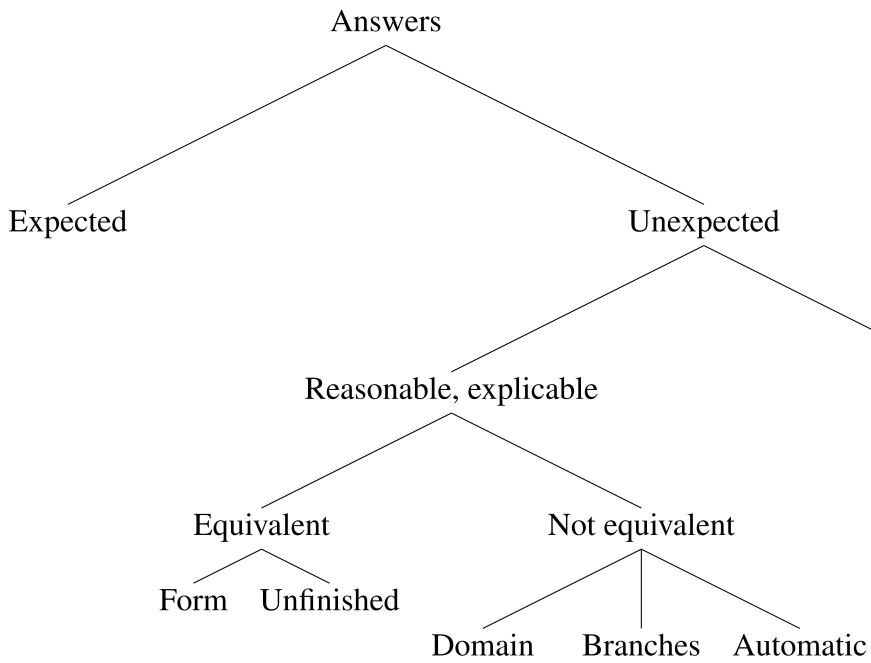

Figure 1: types of answers

The last subtypes are divided further. Hence we use the notation *Type(subtype)*, e.g., *Domain(R/C)*, *Unfinished(log)*. All of them are discussed in section 5.

4 Equations

4.1 Introduction

Equations form an essential part of school algebra. The main types of school equations are discussed in the paper. However, other equations (e.g., Diophantine equations) could be included in the curriculum in specific regions. The curricula of different countries and school types include different amounts of equations. The order of study and names of equation types can vary as well, e.g., radical equation – irrational equation, literal equation – equation with parameter, absolute-value equation – equations with absolute value – equations containing absolute value.

The following subsections include a brief introduction of every type (and subtype) of equations. This is followed by a discussion of the interesting phenomena. The test examples were selected with the intention to find simple non-trivial examples, characteristic of the particular (sub)type. Some were taken directly from a textbook; others were simplified from textbook examples. There are also examples from previous papers. The actual number of tested examples was higher than the number of examples presented in the paper.

4.2 Linear equations

The first equations we study are linear equations. Linear equations are the first equations learned in school and may seem to be quite trivial for both students and CAS. Still, some important phenomena appear here. One could determine three subtypes of linear equations: equation that has one solution, equation that is true for all numbers, and equation that has no solutions.

The only solution could be (at least at the school level where linear equations are introduced) an integer or a fractional number. There are no problems with the integer, but one could notice two points in case of a fractional answer. In case of $3x + 2 = 0$, all CAS give $-2/3$ (or $-\frac{2}{3}$). In case of the equation $2x = 3$, only Sage gives 1.5, the others $3/2$ (or $\frac{3}{2}$). Should the solution be a common fraction or a decimal fraction? Although students should be familiar with both forms, the particular form selected by the program could be slightly unexpected. We denote this issue as *Form(fraction)*. The second issue is more country-dependent. Namely, mixed numbers are used in some countries and, instead of $3/2$, the answer $1\frac{1}{2}$ is expected. The CAS do not support mixed numbers. We denote the issue as *Form(mixed)*.

The cases where there are no solutions or the solution set consists of all numbers lead to the notation issue denoted as *Form(all/empty)*. The CAS use quite different approaches in this case:

- $[x = x]$, $[x == r1]$, $0 = 0$, 'All values of x are solutions'
- [ ], 'No solutions exist', 'Warning, difficulty. It is not possible to find a result or solution'

4.3 Quadratic equations

Quadratic equations could be classified by the (manual) solution process and by the number of real solutions. It is notable that different countries seem to prefer slightly different solutions. For example, solving of quadratic equation by completing the square is popular in several countries, but almost not used at all in others.

Incomplete quadratic equations in the form $ax^2+bx = 0$ (like $10x^2+17x = 0$) are solved without new problems in all CAS. (The issues of presentation of fractions are still present.) The equations in the form $ax^2 + c = 0$ (like $2x^2 - 8 = 0$) raise the issue of sign $\pm$ (*Form($\pm$)*), WolframAlpha presents ± 2 as the answer to the equation $2x^2 - 8 = 0$, others present the solutions separately.

A quadratic equation may have 2, 1 (actually two equal) or 0 real solutions. All of them could provide unexpected answers. In case of 2 solutions, the CAS could vary in the order of terms, etc. For example, one of the answers to the equation $2x^2 - 4x - 5 = 0$ could be presented by different CAS as $-\frac{\sqrt{14}}{2} + 1$ (see Figure 2), $-\frac{\sqrt{14}-2}{2}$ (see Figure 3), $\frac{-\sqrt{14}+2}{2}$ (see Figure 4), $-\frac{1}{2}\sqrt{14} + 1$ (see Figure 5) or $1 - \sqrt{\frac{7}{2}}$ (see Figure 6). We denote the issue as *Form(radical)*. In case of two equal solutions, there is a question about presentation of multiple solutions (*Branches(mult)*); all CAS present them one at a time – the equation $2x^2 - 2x + 1 = 0$ has the answer $x = 1$ (not $x_1 = 1$, $x_2 = 1$).

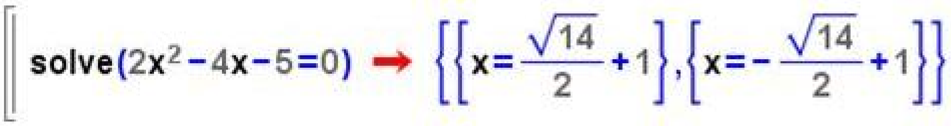

Figure 2: solutions of quadratic equation (WIRIS)

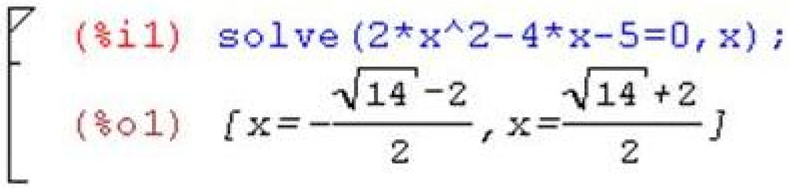

Figure 3: solutions of quadratic equation (Maxima)

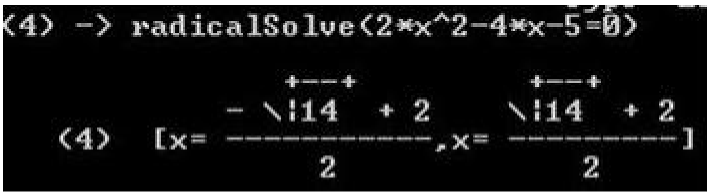

Figure 4: solutions of quadratic equation (Axiom)

The most essential issue related to quadratic equation is presentation of the answer when there are no real solutions (but there are complex solutions, of course). WIRIS gives an empty set { }, Maxima, Sage and WolframAlpha give the complex solutions that include an imaginary unit, the answer of Axiom has a negative number under square root in solutions. We denote the issue as *Domain(C)*.

4.4 Fractional equations

Fractional equations introduce extraneous solutions. One could perform all steps (excl. checking) of the solution algorithm correctly, but the solution could be wrong. There are two possible strategies for managing this case. The 'forbidden' values could be determined in the beginning of solving or the potential answer could be checked when solving is finished. The CAS give correct answers in case of ordinary fractional equations, like $\frac{1}{x} = 0$, $\frac{x-1}{x} = 1$, $\frac{3x-1}{x} - 2 = 0$, $\frac{x-2}{x-1} + 1 = \frac{x-3}{2x-2}$, $\frac{2}{x} - x = 1$ or $\frac{x+1}{x-1} = \frac{2}{x^2-x}$.

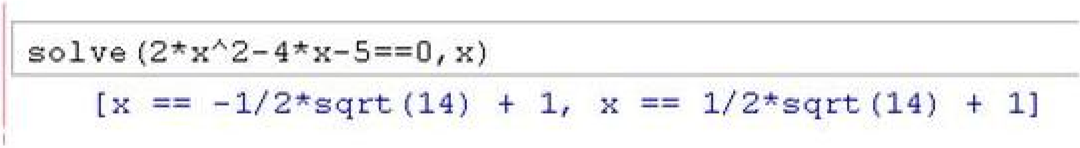

Figure 5: solutions of quadratic equation (Sage)

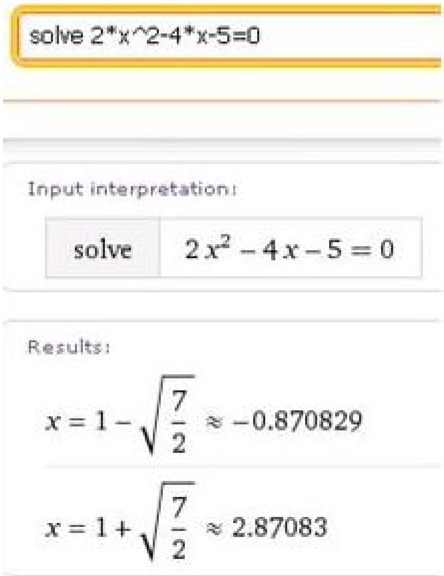

Figure 6: solutions of quadratic equation (WolframAlpha)

One could lure CAS with the somewhat artificial equations $\frac{x \cdot x}{x} = 0$ and $\frac{1}{x} = \frac{1}{x}$. All CAS give disputable answers: the answer 0 to the first equation and all values to the other equation. This is probably related to automatic simplification the equation is automatically transformed to the (standard) form and indeterminacy is cancelled. We denote this issue as *Automatic(indet)*.

4.5 Equations that contain an absolute value

Equations that contain an absolute value could be presented to students after linear equations or much later or not at all. Solving of such equations is complicated and some textbooks deal only with simple examples, solvable by definition.

The first question related to CAS is how to input an absolute value. WolframAlpha understands the mark | | (typed from keyboard), WIRIS has a special button in palette, but the most common approach is to use the function *abs( )*. We denote the issue as *Input(abs)*.

Unlike with the previous types, CAS have quite different efficiencies in case of the equations that contain absolute values. It should be noted that absolute value is also a theoretically complicated topic in computer algebra. WIRIS and WolframAlpha cope well with all examples: $|x| = 3$, $|x| = -3$, $|x+2| = 1$, $|x-3| = |x+2|$, $|x-3| = |3-x|$, $|x+2| - |x| = x-3$, $|x^2-1| = -2x$, $|x^2-x|+3x = 5$,

$||x^3 - \sqrt{x+1}| - 3| = x^3 + \sqrt{x+1} - 7$.

Axiom gives the answers when $\sqrt{(f(x))^2}$ is used instead of $|f(x)|$, e.g., $\sqrt{(x+2)^2} = 1$. We denote this issue as *Input(sqrt(^2))*. However, there will be the extraneous solutions in case of $|x^2 - 1| = -2x$, $|x^2 - x| + 3x = 5$, and in case of $||x^3 - \sqrt{x+1}| - 3| = x^3 + \sqrt{x+1} - 7$ the answer did not appear in reasonable time. Maxima and Sage provide the answer in response to numerical command *find_root*. We denote this issue as *Input (numerical)*.

4.6 Irrational equations

There are several subtypes of irrational equations in school mathematics. Some of them are solvable simply by raising to power (maybe more than once) and then solving a linear or a quadratic equation. Sometimes, the equation is algebraic equation with respect to the radical expression. The checking of solutions is essential part of solving.

In case of CAS, the first question is again about input. While square roots are easily to enter (special button or function *sqrt()*), the other roots (e.g., $\sqrt[3]{\ }$, $\sqrt[6]{\ }$) could be problematic (see *Input(radical)*).

All CAS provide correct solutions to simpler irrational equations with only one radical on the one side and a number on the other, like $\sqrt{x} = 0$, $\sqrt{x} = 2$, $\sqrt{x+1} = 2$ and $\sqrt{x^2 - 3x} = 2$. Additionally, all CAS can solve $\sqrt{7 - \sqrt{x-3}} = 2$ and $\frac{1}{\sqrt{x}} = \frac{1}{2}$.

In case of the most irrational equations (like $\sqrt{2x} = \sqrt{x+1}$, $\sqrt{x} = x$, $x - 2 = \sqrt{x}$, $x - \sqrt{25 - x^2} = 1$, $\sqrt{x+6} - \sqrt{x-1} = 1$, $\sqrt{2x-1} - \sqrt{x-4} = 2$ and $\sqrt{x+2} - \sqrt{x-2} = \sqrt{3x+2}$) Axiom, WIRIS and WolframAlpha offer correct answers, while Maxima and Sage do not solve symbolically but solve numerically with the *find_root* command (*Input(numerical)*). Axiom, Sage, WIRIS and WolframAlpha solve $\sqrt{x^2} = 2$ correctly, Maxima gives $|x| = 2$ as a reaction (but solves numerically with *find_root*). In case of an empty set of solutions ($\sqrt{x} = -2$, $\sqrt{x+3} = -2$ and $\sqrt{3x+1} = \sqrt{x} - 1$), Axiom, WIRIS and WolframAlpha give correct reaction. WIRIS and WolframAlpha are correct with $\sqrt{3x+4} + \sqrt{x} = -3$, but Axiom gives a faulty answer.

Axiom and WolframAlpha present a real solution that is considered as extraneous in school: -1 in case of $\sqrt{2x} = \sqrt{x-1}$ and $-\frac{27}{7}$ (in addition to 3) in case of $\sqrt{2x+6} + \sqrt{x-3} = 2\sqrt{x}$. We denote this issue as *Domain(R/C)*.

In case of $\sqrt[3]{x+45} - \sqrt[3]{x-16} = 1$, WIRIS gives both solutions (80 and -109), Axiom and WolframAlpha give only 80. When -109 is substituted to the left side of the equation, the CAS give $-\sqrt[3]{-1}$ as a result. Is it equal to 1? We denote this issue as *Form(cbrt)*. Maxima and Sage do not solve it symbolically but solve numerically with *find_root*. WIRIS and WolframAlpha solve $\sqrt[3]{x+1} + 2\sqrt[6]{x+1} = 3$ correctly. Axiom gives, in addition to 0, a strange faulty solution. Maxima and Sage do not solve it symbolically but solve numerically with *find_root*.

4.7 Exponential equations

Some useful ideas for solving exponential equations are introduced in schools. One of them is to notice whether bases are equal (or transformable to equal). Another idea is to use logarithms. There are equations that are linear or quadratic equation with respect to the term a^x and are also more complicated.

The first exponential equation observed is $2^x = 2^3$ (same as $2^x = 8$). WolframAlpha gives a correct answer 3. WIRIS gives 3.0 which is correct, but seems to be numerical. We denote this as

Form(1.). Axiom, Maxima and Sage give $\frac{\log(8)}{\log(2)}$ as a solution. It seems to be somewhat unfinished and we denote this as *Unfinished(log)*. The same situation occurs in case of $\sqrt{3^x} = 9$, where the left side of the equation is a little bit trickier, but the right side is still a number. When there is an exponential term on the right side as well, $2^{x+1} = 2^{2x}$, Axiom gives the exact answer but Maxima and Sage do not solve symbolically but give the solution numerically with the help of *find_root* (*Input(numerical)*).

If the same base is somewhat hidden ($3^{2x-1} = 27^x$ or $5^{1-x} + \left(\frac{1}{5}\right)^{x-2} + 25^{-\frac{x}{2}} = 155$), Axiom does not solve, but if the first step is made manually ($3^{2x-1} = 3^{3x}$ and $\frac{5}{5^x} + \frac{25}{5^x} + \frac{1}{5^x} = 155$), the CAS can solve it. In case of equations that are a linear or quadratic equation with respect to the term a^x ($3^{x+2} - 3^{x+1} + 3^x = 21$, $2^{2x} - 8 \cdot 2^x + 16 = 0$ and $3^{2x-1} - 3^{x-1} - 2 = 0$), WolframAlpha gives a correct answer, WIRIS gives *Form(1.)* answer, Axiom gives *Unfinished(log)* answer, and Maxima and Sage give a numerical answer with the command *find_root*.

WolframAlpha and WIRIS give expected answers to equation $\frac{3^x+3^{-x}}{3^x-3^{-x}} = 2$, $1/2$ and 0.5, respectively. Maxima and Sage give $1/2$ and $\frac{\log(-\sqrt{3})}{\log(3)}$ (*Form(log(−1))*), Axiom gives $\frac{\log(\sqrt{3})}{\log(3)}$ and $\frac{\log(-\sqrt{3})}{\log(3)}$ (*Unfinished(log)*, *Unfinished(log(−1))*). It is notable that WolframAlpha also offers the complex answer (see Figure 7, denoted as *Branches(Cn)*, *Domain(C)*). In case of 'unlikeable' answers (like $3^x = 5$, $3^{2x+1} = 15$ and $e^x = 30$), WIRIS gives *Form(1.)* answer (like 0.73249) and four other CAS *Unfinished(log)* answer, WolframAlpha also gives a numerical value ($\frac{\log(5)}{\log(9)} \approx 0.732487$).

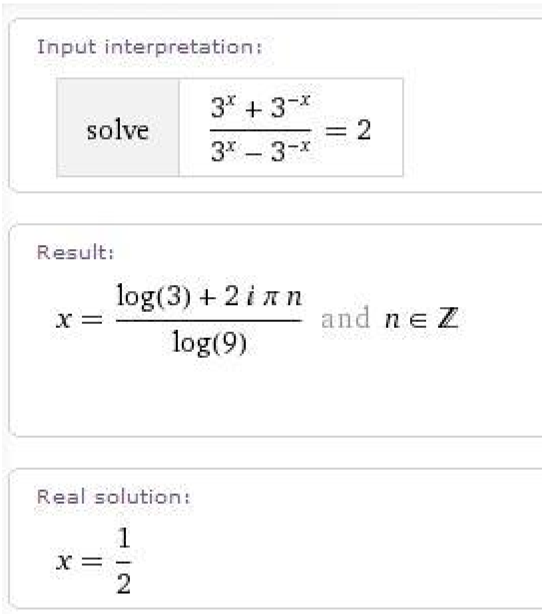

Figure 7: periodic solution to exponential equation (WolframAlpha)

When the number is in decimal form in the equation $10^{-x} = 0.0347$, manual transformation to $10^{-x} = \frac{347}{10000}$ is needed in Axiom (*Input(fraction)*).

WIRIS gives a warning message in case of equations $2^x = -4$, $e^x + 1 = 0$ that have no real solutions. WolframAlpha gives all sets of complex solutions (like $\frac{\log(4)+i\pi(2n+1)}{\log(2)}$ or $i(2\pi n + \pi$) (denoted as *Branches(Cn)*, *Domain(C)*); Sage gives one complex solution (like $i\pi + \frac{\log(4)}{\log(2)}$) (denoted

Equation	Answer, Remark	CAS	Phenomenon
$\log_2 x = 4.5$	$e^{\frac{9\log(2)}{2}}$	Maxima Sage	*Unfinished(log/e)*
$\log_2 x = 4.5$	Manual transformation 4.5 to 9/2	Axiom	*Input(fraction)*
$\log_2(\log_2 x) = 1$	$e^{2\log(2)}$	Axiom	*Unfinished(log/e)*
$\log_{x+1} 4 = 2$	$e^{\frac{\log(4)}{2}} - 1$	Axiom Maxima Sage	*Unfinished(log/e)*
$\log_{2x}(x^2+3) = 2$, *, **	Answer with command *find_root*	Sage	*Input(numerical)*
$\log_{2x}(x^2+3) = 2$, *, **	[ ]	Axiom	–
$\log_{10}^2(x) - 3log_{10}(x) + 2 = 0$	$e^{2\log(10)}$	Axiom	*Unfinished(log/e)*

* $\log_{10}(x+1) + \log_{10}(x-1) - \log_{10}(2x+5) = \log_{10} 3$
** $\log_{10}^2(100x) - 3\log_{10}^2(10x) - \log_{10}(x) = 14$

Table 1: logarithmic equations

as *Branches(C1)*, *Domain(C)*). Axiom and Maxima give $\frac{\log(-4)}{\log(2)}$ (*Unfinished(log(*-1*))*).

In case of unequal bases (like $2^{2-x} = 3^{2-x}$ and $4^{x+1} - 3^x = 3^{x+2} - 4^x$), WolframAlpha gives the expected answers; Maxima and Sage give numerical answers with the help of *find_root*. WIRIS gives a warning message, and Axiom [ ]. In case of $2 \cdot 4^x - 5 \cdot 6^x + 3 \cdot 9^x = 0$, WolframAlpha gives only a set of complex answers (*Branches(Cn)*, *Domain(C)*) where case $n = 0$ gives the needed real answers.

4.8 Logarithmic equations

Before discussing logarithmic equations, we should consider the function $\log()$. It means $\log_{10}$ in WIRIS, but $\log_e$ in the other CAS used. There are log(b, a) function for the $\log_a b$ in WIRIS, but log(a, b) (or log_a(b)) in WolphramAlpha. In other CAS, the necessary function should be defined (e.g., logv(b,a):=log(b)/log(a)). (See *Input(log)*.)

The main idea in solving logarithmic equations is to transform the equation to the form $\log_a f(x) = c$ or $\log_a f(x) = \log_a g(x)$. We start from equations that already have the required form. All CAS solve the equations $\log_2 x = 4$, $\log_2 x = \log_2 14$; WIRIS gives a decimal answer (e.g., $16.$).

Next examples are included in the Table 1. The issues that occur here are *Input* and *Unfinished*.

The exponential logarithmic equation $x^{\log_{10} x} = 100x$ is solved only by WolframAlpha.

WIRIS and Axiom present a real solution that is considered as extraneous in school: -1 in case of $\ln(2x) = \ln(x-1)$. We denote this issue as *Domain(R/C)*.

The equation $\log_b(x^2) = \log_b(2x-1)$ includes a parameter. WolframAlpha gives the answer $x = 1$ and $\log(b) \neq 0$, while Axiom only gives 1. This issue will be discussed in the section of literal equations. Maxima, Sage and WIRIS do not solve it.

4.9 Trigonometric equations

Trigonometric equations could be included in school curricula and textbooks in very different ways. It could be that only the sine, cosine and tangent are treated. The cotangent could also be included, but secant and cosecant are virtually unknown. The complexity of equations is very variable. The variety also extends to the requirements for solution – sometimes a general solution, sometimes one solution or solutions in specified interval. There could also be a question of unit – radian or degree.

Trigonometric equations could be classified in different ways. In this paper, we distinguish the following subtypes:

- Basic equations
  - 'likeable' answer, like $\sin x = 0$, $\cos x = \frac{\sqrt{3}}{2}$ or $\cot x = -1$
  - 'less likeable' answer, like $\sin x = \frac{1}{10}$, $\sin x = 0.1$
  - impossible (in school), like $\sin x = 2$, $\cos x = 2$
- Advanced equations ('one-function')
  - more complicated argument, like $\cos\left(2x - \frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}$
  - factorization, like $\sin x(1 - \sin x) = 0$
  - quadratic equations, like $\sin^2 x - 2\sin x - 3 = 0$
  - biquadratic equations, like $2\tan^4 3x - 3\tan^2 3x + 1 = 0$
- More advanced ('function-change')
  - change function, like $\tan x + 3\cot x = 4$
  - homogeneous, like $2\sin x - 3\cos x = 0$
  - more complicated, like $2 + \cos^2(2x) = (2 - \sin^2 x)^2$

In case of basic trigonometric equations, one could notice the following phenomena. The number of the solutions could be one (Axiom, Sage) or two (WIRIS), or a general solution could be given (WolframAlpha). Maxima gives one solution but also a warning: 'Some solutions may be lost'. This issue is discussed in the section on *Branches(number of solutions)*. The only solution could be different in different CAS, for example, in case of $\cot = -1$, Axiom gives $\frac{3\pi}{4}$ but Maxima and Sage give $\frac{-\pi}{4}$ (*Branches(choice of solution)*). WIRIS does not solve this equation.

Decimal answers are expected in case of 'less likeable' answers, but they could sometimes also appear in case of an expected 'likeable' answer (e.g., $\sin x = \frac{1}{2}$ in WIRIS) (*Form(fraction)*). It is possible that an inverse function (*arcsin*, *arccos*, *arctan*, *arccot*) is included in the solution (*Form(invtrig)*) (see Figure 8). In case of Axiom, the inverse function appears already in the 'likeable' answer (e.g., Axiom, $\sin x = \frac{1}{2}$). In case of Maxima, Sage and WolframAlpha, the inverse function appears in case of 'less likeable' answers (e.g., $\sin x = \frac{1}{10}$). In case of 'impossible' equations (e.g., $\cos x = 2$), WIRIS does not give solutions (that is reasonable). Other CAS give inverse functions in the solution.

The advanced ('one-function') equations add more complicated answers and sometimes the checking of equivalence could be quite intriguing. Are $x = n\pi$ and $x = (-1)^n\frac{\pi}{2} + n\pi$ as found in textbooks

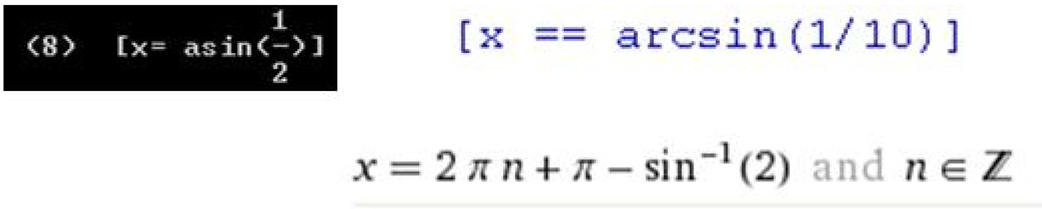

Figure 8: inverse trigonometric functions in answers (Axiom, Sage, WolframAlpha)

equivalent to $x = 2\pi n$, $x = 2\pi n + \pi$ and $x = \frac{1}{2}(4\pi n + \pi)$ as given by WolframAlpha (*Form(periodic)*). Biquadratic trigonometric equation $2\tan^4 3x - 3\tan^2 3x + 1 = 0$ is too complicated for Maxima and Sage, but $2t^4 - 3t^2 + 1 = 0$ is solvable. Therefore, *Input(substitute)*.

Of the more advanced ('function-change') equations, we mention here the equation $2 + \cos^2(2x) = (2 - \sin^2 x)^2$, introduced by [1] as the equation, which has at least three different reasonable ways of solving and each way produces a different-looking answer. Similarly, CAS give different answers (Maxima and Sage do not solve the equation).

4.10 Literal equations

In principle, parameters could be included in every type of equations; one such example was presented in the section on logarithmic equations. Here we focus on linear and quadratic equations. Parameter is very thoroughly discussed in [11]. Drijvers emphasizes that literal equation is basically a different type of equation, because the answer is algebraic expression with letters. The issue is related to obstacle 4 in Drivers list: ***The tendency to accept only numerical solutions and not algebraic solutions.*** *Students often are not satisfied with answers such as $x = \frac{1}{2}s - \frac{1}{2}v$. In the end they want to know what value x stands for. This is called the 'expected answer obstacle'.*

The test set included, e.g., $ax = 1$, $ax = b$, $ax + b = 5$, $x^2 = a$, $V = \pi r^2 h$. One could notice the different forms of the answer, e.g., in case of $ax^2 + bx + c = 0$, WIRIS gives $-\frac{\sqrt{-4ac+b^2}}{2a} + \frac{-b}{2a}$ and $\frac{\sqrt{-4ac+b^2}}{2a} + \frac{-b}{2a}$, Maxima $-\frac{\sqrt{b^2-4ac}+b}{2a}$ and $\frac{\sqrt{b^2-4ac}-b}{2a}$, Sage $-\frac{1}{2} \cdot \frac{b+\sqrt{-4ac+b^2}}{a}$ and $-\frac{1}{2} \cdot \frac{b-\sqrt{-4ac+b^2}}{a}$. The answer of WolframAlpha is shown on Figure 9. (Axiom does not solve this equation.) (*Form(radical).*)

When we look at the WolframAlpha answer, we see different branches. It is very usual that a complete answer to a literal equation has branches (*Branches(literal)*).

Sometimes a branch could be the source of a new problem to be solved by students. For example, WolframAlpha gives to the equation $\frac{3mx-5}{(m+2)(x^2-9)} = \frac{2m+1}{(m+2)(x-3)} - \frac{5}{x+3}$ the answer $x = \frac{21m+38}{6m+9}$ and $3m + 3 \neq 0$ and $9m^3 + 66m^2 + 151m + 110 \neq 0$.

5 Phenomena

5.1 Introduction

More than 25 issues were identified in section 4. There are different possibilities for grouping the issues into types – the following subsections present only one option. In addition to a general description of each type, the subsections present a few brief ideas on the potential didactical use of the

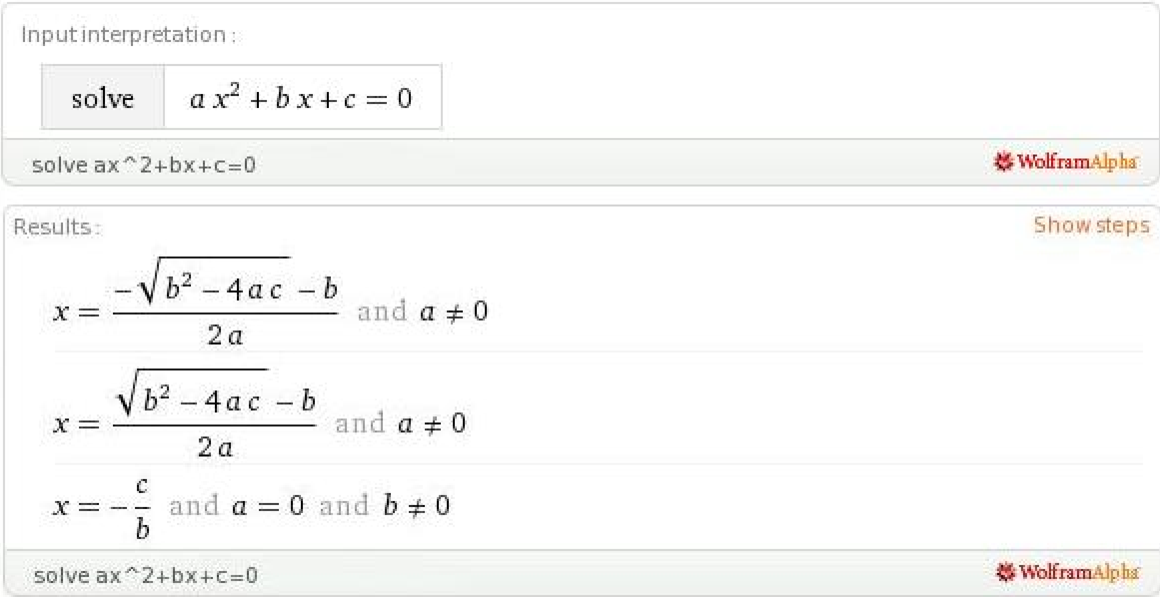

Figure 9: branches in the answer of a literal equation (WolframAlpha)

type. The connections between the type and Drijvers' list of obstacles are presented as well. Drijvers ([11]) introduces the obstacles that students could encounter while working in a computer algebra environment. The paper was based on his experiments. As this paper focuses on a greater number of school equations and a greater number of CAS, the obstacles listed by Drijvers are partially also relevant to phenomena discussed here. The following obstacles are listed in his paper:

1. The difference between the algebraic representations provided by the CAS and those students expect and conceive as simple.
2. The difference between numerical and algebraic calculations and the implicit way the CAS deals with this difference.
3. The flexible conception of variables and parameters that using a CAS requires.
4. The tendency to accept only numerical solutions and no algebraic solutions.
5. The limitations of the CAS, and the difficulty in providing algebraic strategies to help the CAS to overcome these limitations.
6. The inability to decide when and how computer algebra can be useful.
7. The black box character of the CAS.
8. The limited conception of algebraic substitution.
9. The limited conception of algebraic solution.

10. The conception of an expression as a process.

11. The difficult transfer between CAS technique and paper-and-pencil.

12. The difficulty in interpreting the CAS output.

(A CAS may have the option of showing intermediate solution steps. The option is very important in the light of obstacles 7, 11, 12 but is not examined in this paper.)

The structure of the section almost corresponds to the part of Figure 1. The first element of the notation (e.g., *Domain(...)*, *Unfinished(...)*) indicates the title of the subsection. However, we start with the issues that occur before an answer is produced. In these cases, it is necessary to enhance the input line; the command *solve(equation)* is not powerful enough.

5.2 *Input()*

7 phenomena: *Input(abs)*, *Input(sqrt(ˆ2))*, *Input(radical)*, *Input(log)*, *Input(fraction)*, *Input(numerical)*, *Input(substitute)*

There are several cases where simply using the solve-command is not sufficient and some adjustments are needed. They could be related to entering a specific function or operation, for example, absolute value, root, logarithm (*Input(abs)*, *Input(radical)*, *Input(log)*). At first, the different notations and ways of expression could seem obstructive, but they could also be instructive.

Sometimes formulae (like $|a| = \sqrt{a^2}$ (*Input(sqrt(ˆ2))*) or $\log_a b = \frac{\log b}{\log a}$ (*Input(log)*) are required. The use of the formula helps to understand the properties of the operation or function.

The issue is related to obstacle 5 on Drijvers' list: ***The limitations of the CAS, and the difficulty in providing algebraic strategies to help the CAS to overcome these limitations.*** *Sometimes, ..., there is no direct command to perform a task, or the CAS is unable to carry it out without any help from the user. In such cases, cooperation between users' expertise and CAS capacities is needed to find a result.*

The next issue concerns the form of the enterable number (fraction). The use of 4.5 or $9/2$ in equations could give different responses in solving (*Input(fraction)*).

The question of decimal approximation is also important if the CAS does not solve the equation symbolically and numerical solving (for example, with the command *find_root*) is invoked (*Input(numerical)*). This leads to questions about initial values, precision, etc. The importance of numerical calculations and decimal answers could be quite different in different countries. The issue is closely related to obstacle 2 on Drijvers' list: ***The difference between numerical and algebraic calculations and the implicit way the CAS deals with this difference.*** *For many students* $\sqrt{2}$ *is not a real answer: they consider 1.41 as the ultimate result. They do not really understand the difference in status of the two answers: 'still has some algebra in it', whereas 1.41 is purely numerical. The CAS is not always clear about this difference in status.*

It seems to be depend on the country. In several countries, $\sqrt{2}$ is preferred as the ultimate answer.

Sometimes, partial manual solving of the equations helps to produce the answer. For example, instead of $2\tan^4 3x - 3\tan^2 3x + 1 = 0$ (which could be unsolvable for a particular CAS), we will solve $2t^4 - 3t^2 + 1 = 0$. After that, the corresponding trigonometric equation will be solved (*Input(substitute)*). The issue is related to obstacle 8 on Drijvers' list: ***The limited conception of***

algebraic substitution. *Students often think that substitution is limited to 'filling in numerical values'. That conception has to be extended to algebraic substitution of expressions.*

5.3 *Form()*

10 phenomena: *Form(fraction)*, *Form(mixed)*, *Form(all/empty)*, *Form(*$\pm$*)*, *Form(radical)*, *Form(cbrt)*, *Form(1.)*, *Form(log(*-1*))*, *Form(invtrig)*, *Form(periodic)*

This subsection groups together the phenomena where the answer is equivalent to the expected one but different in some way. However, the answers in this subsection seem to be in the ultimate form; the seemingly unfinished answers are treated in the next subsection. (The border between these two classes is quite ambiguous.)

We start with the answers that are equal to the expected one but have a different form. Should we prefer $\frac{3}{2}$, $1\frac{1}{2}$ or 1.5? It is quite country-dependent (*Form(fraction)*, *Form(mixed)*). The situation is slightly different if a decimal answer appears in case of an integer, 1. instead of 1 (*Form(1.)*). The answer could also be expressed by more complicated expressions, like $-\frac{\sqrt{14}}{2}+1$, $-\frac{\sqrt{-4ac+b^2}}{2a}+\frac{-b}{2a}$ (*Form(radical)*) or $x = 2\pi n$, $x = 2\pi n + \pi$ and $x = \frac{1}{2}(4\pi n + \pi)$ (*Form(periodic)*), and the answer could differ from the expected one. The handling of $-\sqrt[3]{-1}$ (*Form(cbrt)*) can also be placed in this group of phenomena.

The main message is that the solution could be in a different but still correct form. The checking of equivalence could be easy or not so easy but instructive. The issue is related to obstacle 1 on Drijvers' list: ***The difference between the algebraic representations provided by the CAS and those students expect and conceive as 'simple'.*** *Recognizing equivalent expressions is a central issue in algebra, and still is when working in a computer algebra environment.*

Sometimes the response of a CAS could contain symbols that are unfamiliar to student. If a symbol (like $\pm$) is included in curriculum (but may be presented in a later chapter) the introduction by CAS is suitable (*Form(*$\pm$*)*). If the symbol is not in the curriculum of the school, the introduction could be problematic but still instructive. There are also some CAS-dependent notation questions (like $[x = x]$ or [ ] in case of equations where the solution set includes all numbers or is empty) (*Form(all/empty)*). Understanding the different error or warning messages is also a part of understanding the notation.

A somewhat transitional issue between this and the next subsection is the appearance of inverse trigonometric functions in the solution (*Form(invtrig)*).

5.4 *Unfinished()*

3 phenomena: *Unfinished(log)*, *Unfinished(log(*-1*))*, *Unfinished(log/e)*

This subsection discusses the solutions that seem to be unfinished, so that almost every user would like to simplify them. The subsection is somewhat parallel to the Input subsection where one should make (manual) steps before solving. This type only contains logarithm-related examples. There are cases (like $\frac{\log(8)}{\log(2)}$ (*Unfinished(log)*) and $e^{2\log(2)}$ (*Unfinished(log/e)*)) where only the final step seems to be needed. The negative argument of logarithm ($\frac{\log(-\sqrt{3})}{\log(3)}$ (*Unfinished(log(-1))*)) leads to the question of domains.

5.5 *Domain()*

2 phenomena: *Domain(C)*, *Domain(R/C)*

Complex numbers are included in school curricula in some countries but not in others. If complex numbers are not included, complex solutions would be unexpected. The solution could explicitly include an imaginary unit or a negative number under square root or as an argument of logarithm (*Domain(C)*). Nevertheless, the introduction of a complex solution could be instructive, at least by indicating that there are more numbers than we use in school.

It is somewhat disputable what would be the correct answer for equations like $\sqrt{2x} = \sqrt{x-1}$ or $\ln(2x) = \ln(x-1)$. On the one hand, -1 is, of course, a real solution. On the other hand, it is not appropriate when operating with real numbers only, since a negative number appears under the square root signs and as an argument for $\ln$ (*Domain(R/C)*). The topic is more thoroughly discussed in [29].

5.6 *Branches()*

6 phenomena: *Branches(mult)*, *Branches(Cn)*, *Branches(C1)*, *Branches(number of solutions)*, *Branches(choice of solution)*, *Branches(literal)*

In many cases when solving an equation, the solution is separable into branches in some manner. [27] introduces a notation for evaluation of branching diversities (EBD) (e.g., $CAS < SCH = MATH$) where CAS refers to the treatment of branches in Computer Algebra Systems, SCH refers to the treatment of branches in school textbooks, and MATH refers to a mathematically branch-complete solution. The equality sign (=) indicates that branches are similarly presented, the sign $<$ shows that the second treatment is more complete.

In case of two equal solutions, some textbooks say that there are two equal roots (SCH(2)), some say one real root (a double root) (SCH(1)), and some say just one real solution (SCH(1)). As all CAS present them only one at a time (at least by default), the possible EBDs are $CAS(1) = SCH(1) < MATH(2)$ or $CAS(1) < SCH(2) = MATH(2)$ ((*Branches(mult)*)).

There are different sources for branching in case of trigonometric equations – periodicity and the families of solutions. The textbooks may or may not provide general solutions, and the same applies to the CAS. Therefore, different EBDs are possible ($CAS < SCH = MATH$, $SCH < CAS = MATH$, $CAS = SCH < MATH$, $CAS = SCH = MATH$) (*Branches(number of solutions)*).

The choice of solution (like $\frac{3\pi}{4}$ or $\frac{-\pi}{4}$) is not expressible by EBD and it is close to the *Form()* phenomena (*Branches(choice of solution)*).

The topic of literal equation is a classic branching topic (*Branches(literal)*). The behaviour of CAS is criticized in [4]. In case of $ax = b$ L. Bernardin says: *When asked to solve with respect to x, all the systems returned the solution* $x = \frac{b}{a}$ *even when this answer is obviously not correct for* $a = 0$. He notes that there may be different commands (e.g., *Reduce* in Mathematica) that work better. Nowadays, at least one system gives the branches by default. If we denote the 'main-branch-approach' as 1 and the 'all-branch-approach' as *all* and, considering that both the textbooks and CAS may use both approaches, there are many possible EBDs: $CAS(1) = SCH(1) < MATH(all)$, $CAS(all) = SCH(all) = MATH(all)$, $SCH(1) < CAS(all) = MATH(all)$ or $CAS(1) < SCH(all) = MATH(all)$.

In addition, the answers in case of complex numbers could include branch issues (*Branches(Cn)*, *Branches(C1)*). However, it is not an issue of the school-level and we will skip it here.

5.7 *Automatic()*

1 phenomenon: *Automatic(indet)*

In case of the solving the somewhat artificial equations $\frac{x \cdot x}{x} = 0$ and $\frac{1}{x} = \frac{1}{x}$, the equations are automatically transformed to the standard form and indeterminacy is cancelled (*Automatic(indet)*). All CAS give disputable answers: 0 in case of the first equation and 'all values' in case of the other. This is probably related to automatic simplification (the topic is discussed in [25]).

6 Conclusion

In a large number of cases when solving school equations (from linear and quadratic to trigonometric and literal) with Computer Algebra Systems, the system gives the answer that is expected by the student or teacher. However, occasionally, this situation also reveals certain phenomena, in which the answer is somewhat different (unexpected). The paper focused on reasonable unexpected answers answers that are not mistakes, but are formulated according to standards differing from school standards. The goal was to classify and map such answers, not to criticize or rank any particular CAS.

More than 120 different equations were solved in five Computer Algebra Systems (Axiom, Maxima, Sage, Wiris and WolframAlpha). The paper identifies 29 types of phenomena with unexpected answers, grouping them into categories and supplying each with a brief introduction. There are separate categories for entering the command and the equation, form of the answer, unfinished solutions, domain issues, branching and automatic simplification. Relations with the items on Drijvers list ([11]) are expressed. Other ways of grouping are certainly possible, but mapping of the issues has been carried out and a (possibly preliminary) classification has been proposed in this paper.

It should be noted that the current study was based only on a limited number of CAS and textbooks. The real instructive value of the phenomena could be determined in a study that involves teachers and students. It seems that the most promising topics for study are the equivalence of answers, the domain issues and branching

7 Acknowledgements

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Software

1. Axiom
2. Maxima
3. Sage
4. WIRIS
5. WolframAlpha

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